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Conjugate Points and Second Order Systems

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1. INTRODUCTION

Consider the linear vector differential equation

$$x'' + A(t)x = 0, \tag{1.1}$$

where $A(t)$ is a continuous $n \times n$ matrix. Throughout this paper, we assume the elements of $A(t)$ to be nonnegative. Our main purpose is to establish a rather basic property of the conjugate points for (1.1). We also establish an implication, involving oscillation, between the scalar and the vector versions of (1.1). A corollary of this result, which, using one of the earlier established results, is quite easy to prove, generalizes the well-known Leighton–Wintner (see [12, 18]) oscillation criterion for the case where $A(t)$ is a nonnegative scalar function.

Recall that a number b is said to be a *right conjugate point* of a number a relative to (1.1) if $b > a$ and if there exists a nontrivial solution of (1.1) which vanishes at a and at b. A number b is called the *first right conjugate point* of a if it is a right conjugate point of a and if there is no number c, $a < c < b$, such that c is also a right conjugate point of a. A left conjugate point and the first left conjugate are defined similarly. It is easy to verify (see [6]) that if a has a right (left) conjugate point then it has a first right (left) conjugate point.

We shall prove that *if b is the first right conjugate point of a then a is the first left conjugate point of b*. The proof of this result seems to be nontrivial. Our proof is based on an argument using induction and several previously established results. The solution to this problem is much simpler and known

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(see [9]) for the case where $A(t)$ is symmetric, in which case variational tools, that seem to be of no use in the general case, are readily available. In fact, a simple and independent proof may be given as follows:

Assume that b is the first right conjugate point of a but a is not the first left conjugate point of b. Then there exists a nontrivial solution $x(t)$ of (1.1) such that $x(b) = x(c) = 0$, where c is some number such that $a < c < b$. Let $A[a, b]$ denote the set of absolutely continuous R^n-valued functions $h(t)$ on $[a, b]$ such that $|h'| \in L^2[a, b]$ and $h(a) = h(b) = 0$. Let $J[h; a, b]$ be the functional defined by

$$J[h; a, b] = \int_a^b (\langle h', h'\rangle - \langle Ah, h\rangle)\, dt$$

over the set $A[a, b]$ of admissible functions. Multiplying the equation

$$x'' + A(t)x = 0$$

by $-x$ and integrating from c to b, we see that $J[x; c, b] = 0$. Now, let $z(t)$ be defined as

$$\begin{aligned} z(t) &= x(t) \qquad \text{if} \quad c \leqslant t \leqslant b \\ &= 0(t) \qquad \text{if} \quad a \leqslant t \leqslant c. \end{aligned}$$

Then, clearly $z \in A[a, b]$ and $J[z; a, b] = 0$. Therefore, $z(t)$ is a solution of (1.1) satisfying $z(a) = z(b) = 0$, since z affords a minimum to J for the class $A[a, b]$ of admissible functions. But this contradicts the assumption that b is the first right conjugate point of a, since $z(a) = z(c) = 0$.

2. Some Known Results

The proof of our main result employs a number of known results from various sources. For the sake of completeness and the reader's convenience, we state such results in this section.

Lemma 2.1 [1, p. 17]. *Let $A(t) = (a_{ij}(t))$ be an $n \times n$ continuous matrix on $[a, b]$ with $a_{ij}(t) \geqslant 0$. Let $y = \text{col}(y_1, \ldots, y_n)$ be a solution of (1.1) with $y(a) = y(b) = 0$ and $y_i(t) \geqslant 0$ for all t in (a, b) and all i, $i = 1, \ldots, n$. If for some k, $k = 1, \ldots, n$, either (i) $y_k'(a) = 0$, (ii) $y_k'(b) = 0$, or (iii) $y_k(c) = 0$ for some c, $a < c < b$, then $y_k(t) \equiv 0$ on $[a, b]$.*

Theorem 2.2 [2, p. 1138]. *Assume $A(t)$ to be as in Lemma 2.1. If b is the first conjugate point of a relative to (1.1), and if $a \leqslant t_1 < t_2 < b$, then there is no nontrivial solution $y(t)$ of (1.1) with $y(t_1) = y(t_2) = 0$.*

THEOREM 2.3 [2, p. 1138]. *Assume $A(t)$ to be as in Lemma 2.1. If b is the first right conjugate point of a relative to (1.1), then there exists a nontrivial solution $u(t) = \mathrm{col}(u_1, \ldots, u_n)$ of (1.1) such that $u(a) = u(b) = 0$ and $u_k(t) \geqslant 0$, $k = 1, \ldots, n$, and $t \in [a, b]$.*

THEOREM (2.3)′. *Assume $A(t)$ as in Lemma 2.1. If a is the first left conjugate point of b relative to (1.1), then there exists a nontrivial solution $u(t) = \mathrm{col}(u_1, \ldots, u_n)$ of (1.1) such that $u(a) = u(b) = 0$ and $u_i(t) \geqslant 0$, $i = 1, \ldots, n$, and $t \in [a, b]$.*

The proof of Theorem (2.3)′ follows from Theorem 2.3 ad the observation that $-a$ is the first right conjugate point of $-b$ relative to

$$x'' + B(t)x = 0,$$

where $B(t) = A(-t)$.

THEOREM 2.4 (see [16, p. 99]. *Let $A(t) = (a_{ij}(t))$ and $B(t) = (b_{ij}(t))$ be continuous $n \times n$ matrices, defined on $[a, b]$, such that $a_{ij}(t) \geqslant b_{ij}(t) \geqslant 0$ for $r \leqslant i, j \leqslant n$ and $t \in [a, b]$. If b is the first right conjugate point of a relative to*

$$x'' + B(t)x = 0,$$

then the first right conjugate point c of a relative to

$$y'' + A(t)y = 0$$

exists and $c \leqslant b$.

A simple proof of Theorem 2.4 is also given in [4].

THEOREM 2.5 [3, p. 36]. *Let $A(t) = (a_{ij}(t))$ be a continuous $n \times n$ matrix function whose elements are nonnegative on $[a, b)$, where b may equal $+\infty$. Suppose that there exists no nontrivial solution $y(t)$ of*

$$x'' + A(t)x = 0$$

such that $y(a) = y(c) = 0$ whenever $a < c < b$. Then there exists no nontrivial solution $z(t)$ of the above equation such that $z(t_1) = z(t_2) = 0$ if $a < t_1 < t_2 < b$.

PROPOSITION 2.6 (see [4]). *Let $A(t)$ be an $n \times n$ continuous matrix-valued function for $t \geqslant t_0$. Suppose that the first right conjugate point of t_0 relative to*

$$x'' + C(t)x = 0$$

exists. If $C^{T}(T)$ denotes the transpose of $C(t)$, then the first right conjugate point of t_0 relative to the adjoint system

$$z'' + C^{T}(t)\,z(t) = 0$$

exists and is equal to the first right conjugate point of t_0 relative to the first equation.

3. A Basic Property of Conjugate Points

THEOREM 3.1. *Let $A(t) = (a_{ij}(t))$ be an $n \times n$ continuous matrix defined on $[a, b]$, with $a_{ij}(t) \geqslant 0$ for $1 \leqslant i, j \leqslant n$ and $t \in [a, b]$. If b is the first right conjugate point of a relative to* (1.1), *then a is the first left conjugate point of b relative to* (1.1).

Proof. First we prove the theorem under the assumption than (1.1) has a solution $v(t) = \operatorname{col}(v_1, \ldots, v_n)$ such that $v(a) = v(b) = 0$ and $v_i(t) > 0$, $i = 1, \ldots, n$, and $t \in (a, b)$. Now, then, suppose that a is not the first left conjugate point of b. Then there exists a number c, $a < c < b$, such that c is a left conjugate point of b. By Theorem 2.2, b must also be the first right conjugate point of c relative to (1.1), since no nontrivial solution of (1.1) can vanish twice in (a, b). By Proposition 2.6, b is also the first conjugate point of c relative to the equation

$$y'' + A^{T}(t)\,y = 0.$$

Hence, by Theorem 2.3, there exists a nontrivial solution $w(t)$ satisfying

$$w'' + A^{T}(t)\,w = 0,$$

$$w(c) = w(b) = 0,$$

and $w(t) \geqslant 0$ on (c, b). Multiplying the first of the equations

$$v'' + A(t)\,v = 0,$$

$$w'' + A^{T}(t)\,w = 0,$$

by $W^{T}(t)$, the second by $-v^{T}(t)$, and adding, we obtain

$$w^{T}v'' - v^{T}w'' = 0$$

since $w^{T}Av = v^{T}A^{T}w$. It follows that

$$w^{T}(t)\,v'(t) - v^{T}(t)\,w'(t) = \text{constant}.$$

Evaluating the above equation at $t = b$ shows that

$$w^{\mathrm{T}}(t)\, v'(t) - v^{\mathrm{T}}(t)\, w'(t) = 0.$$

Letting $t = c$ in the preceding equation, we have

$$v^{\mathrm{T}}(c)\, w'(c) = 0.$$

Since $v_i(t) > 0$, $i = 1,\ldots, n$, we must have $w'(c) = 0$. But $w(c) = w'(c) = 0$ implies that $w(t) \equiv 0$—contradicting the assumption that $w(t)$ is nontrivial. This contradiction shows that a is the first left conjugate point of b relative to (1.1).

Next we wish to show that a is the first left conjugate point of b relative to (1.1), assuming that (1.1) has no solution $v(t) = \mathrm{col}(v_1,\ldots, v_n)$ satisfying $v(a) = v(b) = 0$, and $v_i(t) > 0$, $i = 1,\ldots, n$, $t \in (a, b)$. Our proof will be based on induction on the $n \times n$ nonnegative matrices $A(t)$. Our theorem certainly holds for $n = 1$, since any two nontrivial solutions having a common zero would be constant multiples of each other. Assume that the theorem holds for all nonnegative continuous $k \times k$ matrices $A(t)$, with $1 \leqslant k < n$. We wish to show that it is true for the $n \times n$ matrix $A(t)$ in (1.1).

By Theorem 2.3, there exists a nontrivial solution $y(t) = \mathrm{col}(y_1,\ldots, y_n)$ of (1.1) satisfying $y(a) = y(b) = 0$, and $y_i(t) \geqslant 0$, $i = 1,\ldots, n$, $t \in (a, b)$. Since we are assuming that not all components of $y(t)$ are strictly positive on (a, b), it follows from Lemma 2.1 that for some k, $k = 1,\ldots, n$, $y_k(t) \equiv 0$ on $[a, b]$. By renumbering the equations and components of $y(t)$, we may assume that $y_j(t) > 0$ on (a, b), $j = 1,\ldots, k$, and $y_j(t) \equiv 0$, $j = k + 1,\ldots, n$. Let us partition the matrix $A(t)$ as

$$A(t) = \left[\begin{array}{c|c} M(t) & N(t) \\ \hline O(t) & P(t) \end{array}\right], \tag{3.1}$$

where M is $k \times k$, N is $(n-k) \times k$, O is $k \times (n-k)$ and P is $(n-k) \times (n-k)$. Let us write $y(t)$ as

$$y = \begin{bmatrix} \hat{y} \\ \hat{o} \end{bmatrix},$$

where $\hat{y}$ consists of the first k components of y and $\hat{o}$ consists of $n-k$ components all of which are zero. It is easy to see that $\hat{y}$ satisfies the equation

$$\hat{y}'' + M(t)\, \hat{y} = 0 \tag{3.2}$$

and that the elements of $O(t)$ are all zero. We note that $\hat{y}(a) = \hat{y}(b) = 0$. We now wish to show that a is the first left conjugate point of b relative to the equation

$$x'' + M(t)x = 0. \tag{3.3}$$

Consider the system

$$w'' + B(t)w = 0, \tag{3.4}$$

where $B(t)$ is the $n \times n$ matrix defined by

$$B(t) = \left[\begin{array}{c|c} M(t) & 0_1 \\ \hline 0_2 & 0_3 \end{array}\right],$$

where $0_1, 0_2, 0_3$ are the zero submatrices of appropriate sizes. It is easy to see that the conjugate points of a relative to (3.3) are the same as those relative to (3.4). By Theorem 2.4, the first right conjugate point of a relative to (1.1) is less than, or equal to, the first right conjugate point of a relative to (3.4). On the other hand, b, which is the first conjugate of a relative to (1.1), is a conjugate point of a relative to (3.3) since $\hat{y}(a) = \hat{y}(b) = 0$, and hence relative to (3.4). Therefore, b is the first right conjugate point of a relative to (3.4), and hence relative to (3.3). Consequently, by our induction hypothesis, a is the first left conjugate point of b relative to (3.3).

Now, let us assume that our theorem is false for the $n \times n$ matrix $A(t)$. Then there exists a nontrivial solution $u(t)$ of (1.1) such that $u(c) = u(b) = 0$ for some number c, $a < c < b$. We partition $u(t)$ as

$$u(t) = \begin{bmatrix} \hat{u}(t) \\ u^*(t) \end{bmatrix},$$

where $\hat{u}$ consists of the first k components of u, and u^* consists of the remaining $n - k$ components of u. Then we can write the system

$$u'' + A(t)u = 0$$

as the two systems

$$\hat{u}'' + M(t)\hat{u} + N(t)u^* = 0, \tag{3.5}$$

$$u^{*''} + P(t)u^* = 0. \tag{3.6}$$

We note that $\hat{u}(c) = \hat{u}(b) = 0 = u^*(c) = u^*(b)$. Clearly, $u^*(t) \not\equiv 0$. For, $u^*(t) \equiv 0$ would imply that $\hat{u}$ is a nontrivial solution of (3.3) vanishing at c and b. But this would contradict the fact that a is the first left conjugate point of b relative to (3.3), which was established above.

Now, we consider the system

$$x'' + P(t)x = 0. \tag{3.7}$$

We have shown that c is a left conjugate point of b relative to (3.7). We now wish to show that every number d, $a < d \leqslant c$, is a left conjugate point of b relative to (3.7). Let d be such a number. Let e be the first right conjugate point of d relative to (1.1). It is impossible to have $e < b$, because it would imply that a nontrivial solution of (1.1) vanishes twice in (a, b)—a contrdiction to Theorem 2.2. We cannot have $e > b$, because it would imply that a nontrivial solution (namely $u(t)$) of (1.1) vanishes twice in (d, e)—again a contradiction to Theorem 2.2 (note that $e > b$ implies that $c \neq d$). Finally, Theorem 2.5 assures us that we cannot have $e = +\infty$. Thus, we have shown that b is the first right conjugate point of d relative to (1.1). Let $v(t)$ be a nontrivial solution of (1.1) such that $v(d) = v(b) = 0$. Now, in order to show that b is the first right conjugate point of d relative to (3.7), we partition v as

$$v = \begin{bmatrix} \hat{v} \\ v^* \end{bmatrix},$$

where $\hat{v}$ denotes the first k components of v, and v^* the remaining. Using the partitioning of $A(t)$ given in (3.1) and substituting $v(t)$ in (1.1), we obtain the system

$$\hat{v}'' + M(t)\hat{v} + N(t)v^* = 0,$$
$$v^{*''} + P(t)v^* = 0,$$

where $\hat{v}(d) = \hat{v}(b) = 0 = v^*(d) = v^*(b)$. The same argument that we gave following equation (3.6), in order to show that $u^*(t) \not\equiv 0$, shows that $v^*(t) \not\equiv 0$. This shows that d is a left conjugate point of b relative to (3.7). Consider the differential equation

$$x'' + C(t)x = 0, \tag{3.8}$$

where C is the $n \times n$ matrix given by

$$C = \left[\begin{array}{c|c} 0_1 & 0_2 \\ \hline 0_3 & P(t) \end{array}\right]$$

and where 0_1, 0_2, and 0_3 are zero submatrices of appropriate sizes. Again, we note that the conjugate points of d relative to (3.7) are the same as those relative to (3.8). Making a comparison of $A(t)$ and $C(t)$, and using Theorem 2.4, an argument similar to the one following equation (3.4) shows that b is the first right conjugate point of a relative to (3.8). We have thus shown that for every number d, $a < d \leqslant c$, b is the first right conjugate point of d relative to (3.7). But this means, by our induction hypothesis, that every such number d is the first left conjugate point of b relative to (3.7), which is absurd. This contradiction shows that no nontrivial solution $u(t)$ of (1.1) can exist satisfying $u(c) = u(b) = 0$, $a < c < b$. The proof is now complete.

4. A Theorem on Oscillation

In this section we give a sufficient condition for oscillation of equation (1.1). As in [5], we say (1.1) is oscillatory on $[a, \infty)$ if for each number T, $T \geqslant a$, there exist numbers α and β, with α, $\beta \geqslant T$, and a nontrivial solution $v(t)$ of (1.1) such that $v(\alpha) = v(\beta) = 0$. Our theorem establishes a relationship between the oscillation of scalar equations and vector equations. Similar relationships have been studied in [7, 10, 11, and 14]. However, most of these studies involved selfadjoint systems.

THEOREM 4.1. *Let $A(t) = (a_{ij}(t))$ be a continuous $n \times n$ matrix on $[a, \infty)$, with $a_{ij}(t) \geqslant 0$. If for some pair (i, j), $1 \leqslant i, j \leqslant n$, the scalar equation*

$$x'' + a_{ij}(t)x = 0 \tag{4.1}$$

is oscillatory on $[a, \infty)$, and if $a_{ji}(t) \geqslant a_{ij}(t)$, then the vector differential equation

$$x'' + A(t)x = 0 \tag{4.2}$$

is oscillatory on $[a, \infty)$.

Proof. Consider the system

$$x'' + B(t)x = 0, \tag{4.3}$$

where $B(t) = (b_{ij}(t))$ is the matrix defined by

$$\begin{aligned} b_{sk}(t) &= a_{ij}(t) \qquad \text{if } (k, s) = (i, j) \text{ or } (k, s) = (j, i) \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

$1 \leqslant k$, $s \leqslant n$. Let $T \geqslant a$, be any number. Then there exists a nontrivial solution $v(t)$ of (4.1) and numbers α and β, with $\alpha, \beta \geqslant T$, such that $v(\alpha) = v(\beta) = 0$. Let $u(t) = \operatorname{col}(u_1, ..., u_n)$ be defined by

$$\begin{aligned} u_k(t) &= v(t) \qquad \text{if } k = i \text{ or } k = j, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

$k = 1, 2, ..., n$. It is easy to verify that $u(t)$ is a nontrivial solution of (4.3) satisfying $u(\alpha) = u(\beta) = 0$. Clearly, $b_{ij}(t) \leqslant a_{ij}(t)$ for $1 \leqslant i, j \leqslant n$. Therefore, by Theorem 2.4, there exists a nontrivial solution $z(t)$ of (4.2) such that $z(\alpha) = z(\gamma) = 0$ for some number γ, $\alpha < \gamma \leqslant \beta$. This completes the proof.

EXAMPLE 1. Consider the system

$$x'' + A(t)x = 0,$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easy to verify that the scalar equation

$$x'' + a_{12}(t)x = 0,$$

where $a_{12}(t) = 1$, is oscillatory, but the above system is not oscillatory. This shows that Theorem 4.1 is false, in general, without the assumption that $a_{ii}(t) \geqslant a_{ij}(t)$.

COROLLARY 1. *Let* $A(t)$ *satisfy the condition of Theorem* 4.1. *Then the system* (4.2) *is oscillatory if*

$$\int_a^\infty a_{ii}(t)\, dt = \infty$$

for some i, $1 \leqslant i \leqslant n$.

Proof. The proof is immediate from Theorem 4.1 and the well-known fact (see [12]) that the scalar equation

$$x'' + a_{ii}(r)x = 0$$

is oscillatory.

The above corollary reduces to the well-known Leighton–Wintner theorem (see [8, 12, 18]) when $n = 1$ in (4.2). For the selfadjoint case, this corollary was established in [14]. Also see [7, 10, 11] for similar results or generalizations.

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